ДЗ 10 1/3

№ 1

$$u_{tt} - u_{xx} = 0 \quad x>0 \quad t>0$$

$$\begin{cases} u|_{t=0} = 0 & \\ u_t|_{t=0} = 0 & x \geq 0 \\ u_x|_{x=0} = \frac{1}{3}\sin 3t & t \geq 0 \end{cases}$$

Замена $u(x,t) = \frac{x}{3}\sin 3t + v(x,t)$

$$\begin{cases} v_{tt} - 3x\sin 3t - v_{xx} = 0 \\ v(x,0) = 0 \\ v_t(x,0) + x\cos 3t|_{t=0} = 0 \\ v_x(0,t) + \frac{1}{3}\sin 3t = \frac{1}{3}\sin 3t \end{cases} \Rightarrow \begin{cases} v_{tt} = v_{xx} + 3x\sin 3t \\ v(x,0) = 0 \\ v_t(x,0) = -x \\ v_x(0,t) = 0 \end{cases}$$

Граничное условие II-го рода => продолжаем чётным образом, x-уже нечётная, так что просто оставляем её

Ответ через ф-ию Грина:

$$v(x,t) = 0 + \frac{1}{2a}\int_{x-at}^{x+at} -\xi\, d\xi + \frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} 3\xi \sin 3\tau\, d\xi = \{a=1\}$$

$$= -\frac{1}{2}\frac{\xi^2}{2}\Big|_{x-}^{x+} + \frac{1}{2}\int_0^t 3\sin 3\tau \frac{\xi^2}{2}\Big|_{x-(t-\tau)}^{x+(t-\tau)} d\tau = \frac{(x-t)^2}{4} - \frac{(x+t)^2}{4} + \frac{1}{2}\int_0^t 3\sin 3\tau \cdot \frac{\xi^2}{2}\Big|_{x-(t-\tau)}^{x+(t-\tau)} d\tau =$$

$$= -xt + 3\int_0^t \sin(3\tau)\, x(t-\tau)\, d\tau = -xt + 3xt\left(-\frac{\cos(3\tau)}{3}\Big|_0^t + \int_0^t \sin(3\tau)(-\tau)\, d\tau\right) =$$

$$= -xt + 3xt\left(-\frac{\cos(3t)}{3} + \frac{1}{3} + \frac{\tau\cos(3\tau)}{3}\Big|_0^t - \int_0^t \frac{\cos(3\tau)}{3}\, d\tau\right) =$$

$$= -xt\cos(3t) + xt\cdot t\cos(3t) - \frac{\sin(3\tau)}{9}\Big|_0^t \cdot 3xt =$$

$$= \cos(3t)\cdot xt(t-1) - \frac{1}{3}\sin(3t)\cdot xt$$

$$u(x,t) = \frac{x}{3}\sin(3t) - \frac{x}{3}\sin(3t)\cdot t + xt\cos(3t)(t-1) =$$

$$= x\cdot(t-1)\left(t\cos(3t) - \frac{1}{3}\sin(3t)\right)$$

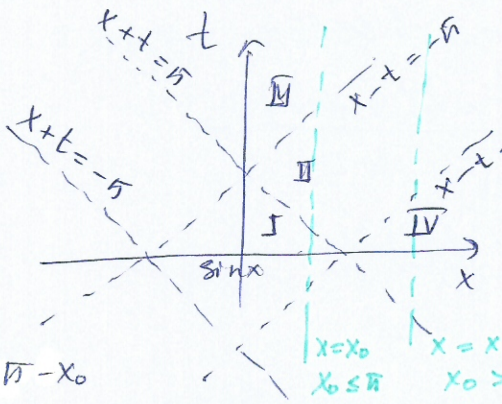
№13

$$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(0,t)=0 \quad t>0 \\ u(x,t) = \begin{cases} \sin x & 0 \le x \le \pi \\ 0 & x \ge \pi \end{cases} \\ u_t(x,0)=0 \end{cases}$$

$a=1$

Граничное условие I рода $\Rightarrow$ продолжаем нечетным образом

$$\varphi(x) = \begin{cases} \sin x & |x| \le \pi \\ 0 & |x| > \pi \end{cases}$$



$$u(x,t) = \frac{\varphi(x-t)+\varphi(x+t)}{2}$$

1) $0 \le x_0 \le \pi$

$$u(x_0,t) = \begin{cases} \frac{\sin(x_0+t)+\sin(x_0-t)}{2}, & 0 \le t < \pi - x_0 \\ \frac{\sin(x_0-t)}{2}, & \pi - x_0 \le t < x_0+\pi \\ 0, & x_0+\pi \le t \end{cases}$$

2) $\pi < x_0$

$$u(x_0,t) = \begin{cases} 0, & 0 \le t < x_0-\pi \\ \frac{\sin(x-t)}{2}, & x_0-\pi \le t < x_0+\pi \\ 0, & x_0+\pi \le t \end{cases}$$

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1) $0 \le t_0 \le \pi$

$$u(x,t_0) = \begin{cases} \sin(x+t_0)+\sin(x-t_0), & 0 \le x < \pi - t_0 \\ \frac{\sin(x-t_0)}{2}, & \pi - t_0 \le x < \pi + t_0 \\ 0, & t_0+\pi \le x \end{cases}$$

2) $\pi < t_0$

$$u(x,t_0) = \begin{cases} 0, & 0 \le x < t_0-\pi \\ \frac{\sin(x-t_0)}{2}, & t_0-\pi \le x < t_0+\pi \\ 0, & t_0+\pi \le x \end{cases}$$

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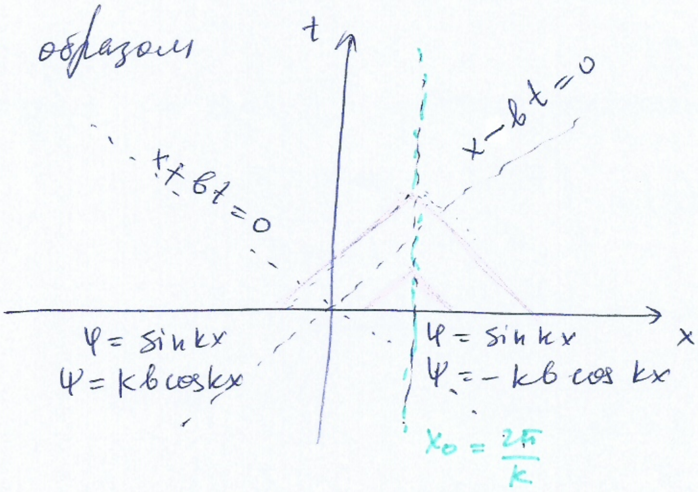
№ 14

$$\begin{cases} u_{tt} = b^2 u_{xx} & x > 0 \\ u(0,t) = 0 \\ u(x,0) = \sin(kx) & k > 0 \\ u_t(x,0) = -k \cdot b \cos(kx) \end{cases}$$

1 Гран условие => прод. нечетным образом

$u(x,0) = \sin(kx) \quad x < 0$

$u_t(x,0) = k \cdot b \cos(kx) \quad x < 0$



$$u(x,t) = \frac{\varphi(x+bt) + \varphi(x-bt)}{2} + \frac{1}{2b}\int_{x-bt}^{x+bt} \psi(\xi)\,d\xi$$

1) $x_0 = \frac{2\pi}{k}$, $\frac{2\pi}{k} \; bt = 0 \Rightarrow t = +\frac{2\pi}{kb}$

$$u\left(\tfrac{2\pi}{k}, t\right) = \begin{cases} \dfrac{\sin\left(k\left(\frac{2\pi}{k}+bt\right)\right) + \sin\left(k\left(\frac{2\pi}{k}-bt\right)\right)}{2} + \dfrac{1}{2b}\displaystyle\int_{\frac{2\pi}{k}-bt}^{\frac{2\pi}{k}+bt} -kb\cos kx\,dx & 0 \le t \le \frac{2\pi}{kb} \\ \dfrac{\overbrace{\sin\left(k\left(\frac{2\pi}{k}+bt\right)\right)}^{\sin kbt} + \overbrace{\sin\left(k\left(\frac{2\pi}{k}-bt\right)\right)}^{\sin(-kbt)}}{2} + \dfrac{1}{2b}\displaystyle\int_{\frac{2\pi}{k}-bt}^{0} kb\cos kx\,dx + \dfrac{1}{2b}\int_{0}^{\frac{2\pi}{k}+bt} -kb\cos kx\,dx & t > \frac{2\pi}{kb} \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left(-\sin kx \Big|_{\frac{2\pi}{k}-bt}^{\frac{2\pi}{k}+bt}\right) & 0 \le t \le \frac{2\pi}{kb} \\ \frac{1}{2}\left(\underbrace{\sin kx \Big|_{\frac{2\pi}{k}-bt}^{0}}_{-\sin(-bt)} - \underbrace{\sin kx \Big|_{0}^{\frac{2\pi}{k}+bt}}_{-\sin bt}\right) & t > \frac{2\pi}{kb} \end{cases} = \begin{cases} -\sin(kbt), & 0 \le t \le \frac{2\pi}{kb} \\ 0, & t > \frac{2\pi}{kb} \end{cases}$$

2) $t_0 = \frac{3\pi}{k \cdot b}$ $\quad -b\frac{3\pi}{kb} + x = 0 \Rightarrow x = \frac{3\pi}{k}$

$$\frac{\sin\left(k\left(x + b \cdot \frac{3\pi}{k \cdot b}\right)\right) + \sin\left(k\left(x - b \cdot \frac{3\pi}{kb}\right)\right)}{2} = \frac{\sin(kx + 3\pi) + \sin(kx - 3\pi)}{2} = 0$$

$$u\left(x, \tfrac{3\pi}{kb}\right) = \begin{cases} \frac{1}{2}\left(-\sin kx \Big|_{x-b\frac{3\pi}{kb}}^{x+b\frac{3\pi}{kb}}\right) & x > \frac{3\pi}{k} \\ \frac{1}{2}\left(\sin kx \Big|_{x-b\frac{3\pi}{kb}}^{0} - \sin kx \Big|_{0}^{x+b\frac{3\pi}{kb}}\right) & 0 \le x \le \frac{3\pi}{k} \end{cases} =$$

$$= \begin{cases} -\sin(xk + \pi) = \sin(xk) & x > \frac{3\pi}{k} \\ 0 & 0 \le x \le \frac{3\pi}{k} \end{cases}$$